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弾性波による構造物調査における数学的貢献について

On the mathematical contribution in the structure investigation by the elastic wave

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あらまし: 弾性波を用いて構造物や空洞などの地下資源の探査を行う際に様々な角度からの数学的な貢献が考えられる. 特に, 弾性波は速さの異なる波が発生し, 地球が等方性の弾性体と仮定するといわゆるP波とS波とよばれる2つの波が発生する. 本講演では, 散乱理論を用いて構造物の凸包を再構成する数学的な手法を紹介するとともに, 弾性波特有のモードの変換とよばれる入射波と反射波の速さが異なる場合の新たな数学的な結果を地下資源探査に応用する方法を提案する.

Abstract:When we analyze the reflection phenomenon for the elastic wave, the one of the most complicated and interesting problems is to study the mode conversion case. For the elastic wave, there are waves of different modes and a remarkable phenomenon called "mode-conversion" which causes serious difficulties. In this paper, by considering the non back-scattering case, we examine the singularities of the scattering kernel for the elastic wave equation with transverse incident waves and derive a new result about the singularities of the scattering kernel.

キーワード: 散乱理論, 弾性方程式, 特異性, モード変換, P-波, S-波.

Key words: scattering theory, elastic wave, singularities, mode-conversion, P-wave, S-wave.

1 Introduction

Let Ω be an exterior domain in $\mathbf{R}^3$ with smooth and compact boundary. We consider the isotropic elastic wave equation with the Dirichlet boundary condition

$$\begin{cases} (\partial_t^2 - L)u(t,x) = 0 & \text{in } \mathbf{R}\times\Omega, \\ u(t,x) = 0 & \text{on } \mathbf{R}\times\partial\Omega, \\ u(0,x) = f_1(x) \qquad \partial_t u(0,x) = f_2(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $u(t,x) = {}^t(u_1,u_2,u_3)$ and $f_i(x) = {}^t(f_{i1}, f_{i2}, f_{i3})(i=1,2)$. Recall that L has the following form:

$$L = \sum_{i,j=1}^{3} a_{ij}\partial_{x_i}\partial_{x_j},$$

where a_{ij} are 3×3 matrices of which (p,q)-entry is expressed by a_{ipjq}. We say that the elastic medium Ω is isotropic, if a_{ipjq} is given by

$$a_{ipjq} = \lambda\delta_{ip}\delta_{jq} \;+\; \mu(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp}),$$

where λ, μ are Lame's constants satisfying the following inequalities:

$$\lambda + \frac{2}{3}\mu > 0 \ , \ \mu > 0.$$

Under the assumption that the elastic medium Ω is isotropic, Yamamoto [13] and Shibata-Soga [8] have formulated a scattering theory which is analogous to the theory of Lax-Phillips [5]. Let $k_-(s,\omega)$ and $k_+(s,\omega) \in L^2(\mathbf{R}\times S^2)$ denote the incoming and outgoing translation representations of an initial data $f =^t (f_1, f_2)$ respectively (see [5]). Recall that the scattering operator S is the mapping

$$S \quad : \quad k_-(s,\omega) \quad \longmapsto \quad k_+(s,\omega).$$

The scattering operator S admits a representation:

$$(Sk_-)(s,\theta) = \iint_{\mathbf{R}\times S^2} S(s-\tilde{s},\theta,\omega)k_-(\tilde{s},\omega)d\tilde{s}d\omega$$

with a distribution kernel $S(s,\theta,\omega)$ called the scattering kernel. Majda [6] has obtained a representation formula of the scattering kernel $S(s,\theta,\omega)$ for the scalar-valued case. This representation formula is very effective to investigate inverse scattering problems (cf. Majda [6], Soga [9], Petkov [7]). For the elastic case, Soga [10] and Kawashita [3] have derived a representation formula of the scattering kernel.

The characteristic matrix $L(\xi)$ of the operator $L(\partial_x)$ has the eigenvalues $C_1^2|\xi|^2$ and $C_2^2|\xi|^2$, where

$$C_1 = (\lambda + 2\mu)^{\frac{1}{2}}, \qquad C_2 = \mu^{\frac{1}{2}}.$$

Let $P_i(\xi)$ be the eigenprojector associated to the eigenvalues $C_i^2|\xi|^2 (i = 1,2)$, where

$$P_1(\xi) = \xi \otimes \xi, \qquad P_2(\xi) = I - P_1(\xi).$$

Then $P_1(\xi)\mathbf{R}^3$ is the space spaned by ξ, and $P_2(\xi)\mathbf{R}^3$ is the orthogonal complement of $P_1(\xi)\mathbf{R}^3$. Associated with the eigenvalues $C_i^2|\xi|^2(i = 1,2)$, there are waves of two different types (modes). The one propagates with the speed C_1, and the other with C_2. Furthermore their amplitudes are longitudinal and transverse to the propagation direction respectively, and therefore these waves are called longitudinal and transverse waves respectively. For elastic waves there is a remarkable phenomenon called "mode-conversion", that is, when longitudinal or transverse incident wave hits the boundary $\partial\Omega$, both longitudinal reflected wave and transverse reflected wave appear. This phenomenon causes serious difficulties in the analysis of singularities of the scattering kernel for the elastic wave equation.

In view of results concerning mode-conversion (cf. Chapter 5 of Achenbach [1] and Theorem 2.1 of Soga [12]), we can expect that corresponding phenomenon occurs for the scattering kernel $S(s,\theta,\omega)$, because in the asymptotic sense the kernel $P_i(\theta)S(C_i^{-\frac{1}{2}}\theta \cdot x - t, \theta, \omega)P_l(\omega)$ expresses the C_i-mode component of the scattered wave in the direction θ for the C_l-mode incident plane wave in the direction ω. In the back-scattering case (i.e $\theta = -\omega$), by Soga [10, 11] we can obtain results of the same type as in Majda [6]. Moreover, in Kawashita-Soga [4], they have derived same results in the mode-conversion case. In Ota [14], by considering the non back scattering case (i.e $\theta \neq \omega$), we have the following results :

Theorem 1.1. *Let $\omega, \theta \in S^2$. Assume that $|\theta + \omega|$ is different from zero and sufficiently small, and $n_{il}(\theta,\omega)$ is a regular direction for $\partial\Omega$. Then we have*

(i) $\text{supp}[P_i(\theta)S(\cdot,\theta,\omega)P_1(\omega)] \subset (-\infty, -r_{i1}(\theta,\omega)]$ $(i = 1,2)$

(ii) $P_i(\theta)S(s,\theta,\omega)P_1(\omega)$ *is singular (not C^∞) at* $s = -r_{i1}(\theta,\omega)(i = 1,2)$.

In Ota [14], we have derived an asymptotic expansion of the scattering kernel in the non back scattering case (cf. Soga [11]). In this paper, by means of this expansion, we investigate the singularities of $P_i(\theta)S(s,\theta,\omega)P_2(\omega)$ and shall show that the leading term of this asymptotic expansion of $P_i(\theta)S(s,\theta,\omega)P_2(\omega)$ don't vanish.

2 Main results

Before giving the main results in the present paper, according to Ota [14], we give several definitions for stating those.

We set $r_{il}(\theta,\omega) := \min_{x\in\partial\Omega} x\cdot n_{il}(\theta,\omega)$, where $n_{il}(\theta,\omega) := -(C_i^{-1}\theta - C_l^{-1}\omega)$. Next, we denote the first hitting points at $\partial\Omega$ by $N_{il}(\theta,\omega) := \{x; n_{il}(\theta,\omega)\cdot x = r_{il}(\theta,\omega)\} \cap \partial\Omega$. Furthermore, we arbitrarily pick a point $a_t \in N_{il}(\theta,\omega)$ and choose a system of orthogonal local coordinates $y = (y', y_3)$, with $y' = (y_1, y_2)$, in $\mathbf{R}^3$ such that $y_3 = (r_{il}(\theta,\omega) - n_{il}(\theta,\omega)\cdot x)|n_{il}(\theta,\omega)|^{-1}$, and that $y = 0$ expresses the reference point a_t. Then Ω is represented by $y_3 > \psi(y')$ in a neighborhood U of a_t, where $\psi(y')$ is a C^∞ function defined in a neighborhood of $y' = 0$.

If the Hessian matrix $H_{\psi(y')}$ of $\psi(y')$ is negative definite at $y' = 0$ for every such picked point, we say that $n_{ij}(\theta,\omega)$ is a regular direction for $\partial\Omega$, which does not depend on the choice of the coordinates $y = (y', y_3)$. If $n_{il}(\theta,\omega)$ is a regular direction, the set $N_{il}(\theta,\omega)$ consists of a finite number of isolated points.

For a distribution $f(s)$ on $\mathbf{R}$ we use the notation

$$f(s) \sim f_0(s) + f_1(s) + \cdots \qquad \text{at } s_0,$$

which means that there exists an integer m and a C^∞ function $\varphi(s)$ with $\varphi(s_0) \neq 0$ such that for every integer $N \geq 0$

$$\varphi(s)\{f(s) - (f_0(s) + \cdots + f_N(s))\} \in H^{m+N}(\mathbf{R}).$$

Then we have

Theorem 2.1. *Let $\omega, \theta \in S^2$. Assume that $|\theta+\omega|$ is different from zero and sufficiently small, and $n_{il}(\theta,\omega)$ is a regular direction for $\partial\Omega$. Then we have*

$P_i(\theta)S(s,\theta,\omega)P_1(\omega)$ is singular (not C^∞) at

$$s = -r_{i1}(\theta,\omega) \qquad (i = 1, 2).$$

3 Asymptotic expansion of the scattering kernel

In order to examine the singularities of $P_i(\theta)S(s,\theta,\omega)P_l(\omega)$, it is useful to know the asymptotic behavior of the scattering kernel. In this section we shall derive an asymptotic expansion of the scattering kernel which plays an essential role in the proof of Theorem 2.1.

In order to derive an expansion of $P_i(\theta)S(s,\theta,\omega)P_l(\omega)$, we review some results in [11]. Let $v_l(t,x;\omega)$ be the solution of the following boundary value problem:

$$\begin{cases} (\partial_t^2 - L)v_l(t,x;\omega) = 0 & \text{in } \mathbf{R}\times\Omega \\ v_l(t,x;\omega) = (2\sqrt{2}\pi)^{-2}C_l^{-\frac{3}{2}}\delta(t - C_l^{-1}\omega\cdot x)P_l(\omega) & \\ & \text{on } \mathbf{R}\times\partial\Omega \\ v_l(t,x;\omega) = 0 & \text{for } t < C_l^{-1}r(\omega) \end{cases} \tag{3.1}$$

where $r(\omega) = \min\limits_{x\in\partial\Omega} x\cdot\omega$. Namely $v_l(t,x;\omega)$ is the scattered wave for the incident wave

$$(2\sqrt{2}\pi)^{-2}C_l^{-\frac{3}{2}}\delta(t - C_l^{-1}\omega\cdot x)P_l(\omega). \tag{3.2}$$

The scattering kernel is represented by means of $v_l(t,x;\omega)$:

$$S(s,\theta,\omega)$$
$$= \sum_{i,j=1}^{2} C_i^{-\frac{3}{2}} \int_{\partial\Omega} \Big\{ P_i(\theta)(\partial_t N v_j)(C_i^{-1}\theta\cdot x - s, x;\omega)$$
$$-C_i^{-1}P_i(\theta)^t(N(\theta\cdot x))(\partial_t^2 v_j)(C_i^{-1}\theta\cdot x - s, x;\omega)\Big\} dS_x \tag{3.3}$$

where $N = \sum_{i,j=1}^{3} a_{ij}\nu_i\partial_{x_j}$ and $\nu = (\nu_1,\nu_2,\nu_3)$ is the unit outer normal to Ω.

By the Hamilton-Jacobi method we have a real-valued C^∞ function $\varphi_l^k(x)(k,l=1,2)$ satisfying

$$\begin{cases} |\nabla\varphi_l^k(x)| = \dfrac{1}{C_k} & \text{in } \Omega\cap U_\epsilon, \\ \varphi_l^k(x) = \dfrac{1}{C_l}\omega\cdot x & \text{on } \partial\Omega\cap U_\epsilon, \\ \dfrac{\partial\varphi_l^k}{\partial\nu}(x) < 0 & \text{on } \partial\Omega\cap U_\epsilon, \end{cases} \tag{3.4}$$

where $U_\epsilon = \{x; |n_{il}(\theta,\omega)\cdot x - r_{il}(\theta,\omega)| < \epsilon\}$ with a small $\epsilon > 0$.

We set

$$\rho_j(t) = \begin{cases} \dfrac{t^{j-1}}{(j-1)!} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \quad \text{when } j = 1,2,\cdots,$$
$$\rho_j(t) = \rho_{j+1}'(t) \qquad \text{when } j = 0,-1,\cdots.$$

Let us note that

$$\rho_0(t) = \delta(t)\ ,$$
$$\rho_{j+1}'(t) = \rho_j(t) \qquad \text{for any integer } j.$$

Lemma 3.1. *Assume that there exists a sufficiently small $\delta > 0$ such that $|\nabla\varphi_l^k{}_{tan}| < \delta$. Then the solution v_l of (3.1) admits the following asymptotic expansion for $t \in R$ sufficiently close to $r_{ij}(\theta,\omega)$*

$$v_l(t,x;\omega) \sim \sum_{k=1}^{2}\sum_{j\geq 0}\rho_j(t-\varphi_l^k(x))u_{lj}^k(x), \tag{3.5}$$

where $u_{lj}^k(x)$ are some C^∞functions defined in $\bar\Omega\cap U_\epsilon$ and $\nabla\varphi_l^k{}_{tan}$ denotes the tangential part to $\partial\Omega$ of $\nabla\varphi_l^k$.

Proof. Combining Theorem 2.1 in Soga [11] and Theorem 1.1 in [12], we can derive the above asymptotic expansion in this case. □

Let $v_l(t,x;\omega)$ be the solution of (3.1). Then $u = v_l P_l(\omega)$ satisfies the equation $(\partial_t^2 - L)u = 0$ in $\mathbf{R}\times\Omega$ and verifying the same boundary condition as v_l. Hence, by the uniqueness of the solutions, we obtain that $v_l(t,x;\omega) = v_l(t,x;\omega)P_l(\omega)$. Moreover combining the representation of the scattering kernel (3.3) and the asymptotic expansion (3.5), we have

$$P_i(\theta)S(s,\theta,\omega)P_l(\omega)$$
$$\sim \sum_{k=1}^{2} C_i^{-\frac{3}{2}} \Bigg[\sum_{j\geq -1}\int_{\partial\Omega}\rho_{j-1}(-s - n_{il}(\theta,\omega)\cdot x)$$
$$\times P_i(\theta)\sum_{p,q=1}^{3} a_{pq}\nu_p(x)\{(-\partial_{x_q}\varphi_l^k(x))u_{lj+1}^k(x)$$
$$+\partial_{x_q}u_{lj}^k(x)\}P_l(\omega)dS_x$$
$$-C_i^{-1}\sum_{j\geq 0}\int_{\partial\Omega}\rho_{j-2}(-s - n_{il}(\theta,\omega)\cdot x)P_i(\theta)$$
$$\sum_{p,q=1}^{3} {}^t a_{pq}\nu_p(x)\theta_q u_{lj}^k(x)P_l(\omega)dS_x\Bigg]. \tag{3.6}$$

For a regular direction $n_{il}(\theta,\omega)$ we have $N_{il}(\theta,\omega) = \{a_1,\cdots,a_M\}$. By using a partition of unity, it is enough to examine the terms whose integrands are supported on a small neighborhood of the reference point $a_t \in N_{il}(\theta,\omega)$. Then we can rewrite the above integrals (3.6) as $\sum_{t=1}^{M} I_t(\theta,\omega)$. Since the analysis of above integrals near each point a_t is same , it is sufficient to study the leading term in $I_t(\theta,\omega)$ for only one a_t, where we may assume $a_t = 0$.

We take an orthonormal frame $\{p_1,p_2,p_3\}$ where $p_3 = -n_{il}(\theta,\omega)|n_{il}(\theta,\omega)|^{-1}$, and choose the local coordinate system $y = (y_1,y_2,y_3)$ such that $x = y_1p_1 + y_2p_2 + y_3p_3$. Let us denote by T the 3×3 orthogonal matrix $T = (t_{pq})$ such that $T(e_j) = p_j (j = 1,2,3)$, where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbf{R}^3$. Then $\partial\Omega$ is represented by $y_3 = \psi(y')$ near 0. Since the equation is isotropic, we have the following result.

Lemma 3.2. *Assume that the elastic medium Ω is isotropic, then we have*

$$TL({}^tT\xi){}^tT = L(\xi) \text{ and } T\sum_{p,q=1}^{3} a_{pq}t_{rp}t_{sq}{}^tT = a_{rs}.$$

Proof. By the isotropicity of the equation,

$$TL({}^tT\xi){}^tT = T\{(\lambda+\mu){}^tT\xi\otimes{}^tT\xi+\mu|T\xi|^2I\}{}^tT = (\lambda+\mu)\xi\otimes\xi+\mu|\xi|^2I = \sum_{r,s=1}^{3} a_{rs}\xi_r\xi_s.$$

On the other hand, a direct computation shows

$$\sum_{p,q,m,n=1}^{3} t_{im}a_{pmqn}t_{rp}t_{sq}t_{jn} = \lambda\delta_{ri}\delta_{sj} + \mu(\delta_{rs}\delta_{ij}+\delta_{rj}\delta_{si}) = a_{risj}.$$

Thus the proof is complete. □

By Lemma 3.2 and an easy computation, we have the following identities:

$$L(\partial_y)u|_{y={}^tTx} = {}^tTL(\partial_x)Tu({}^tTx) \quad \text{for any } x\in\Omega, \tag{3.7}$$

$$(N_yu)({}^tTx) = {}^tTN_xTu({}^tTx) \quad \text{for any } x\in\partial\Omega, \tag{3.8}$$

where $N_x = \sum_{pq=1}^{3} a_{pq}\nu_p\partial_{x_q}$, $N_y = \sum_{pq=1}^{3} a_{pq}\nu_p^*\partial_{y_q}$ and $\nu^*(y) = {}^tT\nu(Ty)$. Then from (3.7), it follows that $\tilde{v}_l(t,y;\tilde{\omega}) := {}^tTv_l(t,Ty;\omega)T$ satisfies the same boundary value problem (3.1) in $\tilde{\Omega} = {}^tT\Omega$ where ω is replaced by $\tilde{\omega} = {}^tT\omega$. Moreover $\tilde{v}_l(t,y;\tilde{\omega})$ admits the following asymptotic expansion:

$$\tilde{v}_l(t,y;\tilde{\omega}) \sim \sum_{k=1}^{2}\sum_{j\geq 0}\rho_j(t-\tilde{\varphi}_l^k(y))\tilde{u}_{lj}^k(y). \tag{3.9}$$

Here $\tilde{u}_{lj}^k(y) := {}^tTu_{lj}^k(Ty)T$ and $\tilde{\varphi}_l^k(y) := \varphi_l^k(Ty)$ which satisfies

$$\begin{cases} |\nabla\tilde{\varphi}_l^k(y)| = \dfrac{1}{C_k} & \text{in } \tilde{\Omega}\cap\tilde{U}_\epsilon, \\ \tilde{\varphi}_l^k(y)|_{y_3=\psi(y')} = \dfrac{1}{C_l}\tilde{\omega}\cdot y & \text{on } \partial\tilde{\Omega}\cap\tilde{U}_\epsilon, \\ \dfrac{\partial\tilde{\varphi}_l^k}{\partial\nu^*}(y)|_{y_3=\psi(y')} < 0 & \text{on } \partial\tilde{\Omega}\cap\tilde{U}_\epsilon, \end{cases} \tag{3.10}$$

where $\tilde{U}_\epsilon = \{y; |n_{il}(\tilde{\theta},\tilde{\omega})\cdot y - r_{il}(\tilde{\theta},\tilde{\omega})| < \epsilon\}$ with a small $\epsilon > 0$.

Since $P_l(\omega) = TP_l(\tilde{\omega}){}^tT$, $P_i(\theta) = TP_i(\tilde{\theta}){}^tT$, by Lemma 3.2, $I_t(\theta,\omega)$ takes the following form:

$$\begin{aligned}
&\sum_{k=1}^{2} C_i^{-\frac{3}{2}}P_i(\theta)\int_{\mathbf{R}^2}\rho_{-2}(-s+|n_{il}(\theta,\omega)|\psi(y')-r_{il}(\theta,\omega))\\
&\quad\times\left\{\sum_{p,q=1}^{3} a_{pq}\nu_p(Ty)(-\partial_{x_q}\varphi_l^k(Ty))\right.\\
&\quad\left.-C_i^{-1}\sum_{p,q=1}^{3}{}^ta_{pq}\nu_p(Ty)\theta_q\right\}u_{l0}^k(Ty)\beta_{-2}(y')P_l(\omega)dy'\\
&+\sum_{j\geq -1}\int\rho_j(-s+|n_{il}(\theta,\omega)|\psi(y')-r_{il}(\theta,\omega))\beta_j(y')dy'
\end{aligned}$$

$$\begin{aligned}
&=\sum_{k=1}^{2} C_i^{-\frac{3}{2}}\int_{\mathbf{R}^2}\rho_{-2}(-s+|n_{il}(\tilde{\theta},\tilde{\omega})|\psi(y')-r_{il}(\tilde{\theta},\tilde{\omega}))\\
&\quad\times T\left\{P_i(\tilde{\theta})\sum_{r,s=1}^{3}\left\{{}^tT\sum_{p,q=1}^{3}a_{pq}t_{pr}t_{qs}T\right\}\right.\\
&\qquad\qquad\times\nu_r^*(y)(-\partial_{y_s}\tilde{\varphi}_l^k(y))\\
&\quad\left.-C_i^{-1}\sum_{r,s=1}^{3}\left\{{}^tT\sum_{p,q=1}^{3}{}^ta_{pq}t_{pr}t_{qs}T\right\}\nu_r^*(y)\tilde{\theta}_s\right\}\\
&\qquad\qquad\times\left\{{}^tTu_{l0}^k(Ty)T\right\}\beta_{-2}(y')P_l(\tilde{\omega}){}^tTdy'\\
&\quad+\sum_{j\geq -1}\int\rho_j(-s+|n_{il}(\tilde{\theta},\tilde{\omega})|\psi(y')-r_{il}(\tilde{\theta},\tilde{\omega}))\\
&\qquad\qquad\times\beta_j(y')dy'
\end{aligned}$$

$$\begin{aligned}
&=\sum_{k=1}^{2} C_i^{-\frac{3}{2}}\int_{\mathbf{R}^2}\rho_{-2}(-s+|n_{il}(\tilde{\theta},\tilde{\omega})|\psi(y')-r_{il}(\tilde{\theta},\tilde{\omega}))\\
&\times T\left[P_i(\tilde{\theta})\left\{\sum_{r,s=1}^{3}a_{rs}\nu_r^*(y)(-\partial_{y_s}\tilde{\varphi}_l^k(y))\right.\right.\\
&\left.\left.-C_i^{-1}\sum_{r,s=1}^{3}{}^ta_{rs}\nu_r^*(y)\tilde{\theta}_s\right\}\tilde{u}_{l0}^k(y)\beta_{-2}(y')P_l(\tilde{\omega})\right]{}^tTdy'\\
&+\sum_{j\geq -1}\int\rho_j(-s+|n_{il}(\tilde{\theta},\tilde{\omega})|\psi(y')-r_{il}(\tilde{\theta},\tilde{\omega}))\beta_j(y')dy',
\end{aligned}$$

where $\beta_j(y')$ are some C^∞ functions supported near $y' = 0$ and $\beta_{-2}(0) = 1$.

Since $n_{il}(\theta,\omega)$ is a regular direction, by the Morse lemma we can take a new system of local coordinates

$\tilde{y}'$ so that $\tilde{y}'=0$ means $y'=0$ and that

$$\psi(y'(\tilde{y}')) = -\frac{1}{2}|\tilde{y}'|^2,$$
$$\det\frac{\partial \tilde{y}'}{\partial y'}(0) = K(a_t)^{-\frac{1}{2}}.$$

We can determine the phase functions $\tilde{\varphi}_l^k$ and the amplitudes $\tilde{u}_{l0}^k$ by the methods in Kawashita [2]. Applying the Taylor expansions to $\nu^*(\tilde{y}), \nabla\tilde{\varphi}_l^k(\tilde{y})|_{y_3=\psi(\tilde{y}')}, \tilde{u}_{l0}^k(\tilde{y})|_{y_3=\psi(\tilde{y}')}$:

$$\begin{aligned} \nu^*(\tilde{y}) &= |n_{il}(\tilde{\theta},\tilde{\omega})|^{-1}n_{il}(\tilde{\theta},\tilde{\omega}) + \cdots, \\ \nabla\tilde{\varphi}_l^k(\tilde{y})|_{y_3=\psi(\tilde{y}')} &= C_l^{-1}\,{}^t(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\kappa}_{kl}) + \cdots, \\ \tilde{u}_{l0}^k(\tilde{y})|_{y_3=\psi(\tilde{y}')} &= (2\sqrt{2}\pi)^{-2}C_l^{-\frac{3}{2}}\tilde{P}_{k,0}^l P_l(\tilde{\omega}) + \cdots, \end{aligned}$$

where $\tilde{y} = (\tilde{y}', y_3)$, $\tilde{\kappa}_{kl} = \sqrt{\tilde{\omega}_3^2 + C_l^2\cdot C_k^{-2} - 1}$ and $(\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\cdot\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')})^{-1}\cdot\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\otimes\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{1,0}^l + O(|\tilde{y}'|)$ and $I - (\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\cdot\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')})^{-1}\cdot\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\otimes\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{2,0}^l + O(|\tilde{y}'|)$. We can rewrite the integrals $I_t(\theta,\omega)$ in the following way:

$$\begin{aligned} &(2\sqrt{2}\pi)^{-2}C_i^{-\frac{3}{2}}C_l^{-\frac{3}{2}} \\ &\times\int_{\mathbf{R}^2}\rho_{-2}(-s-|n_{il}(\tilde{\theta},\tilde{\omega})||\tilde{y}'|^2/2 - r_{il}(\tilde{\theta},\tilde{\omega})) \\ &\qquad\times T\Bigg[P_i(\tilde{\theta})\sum_{k=1}^{2}\Bigg\{\sum_{p=1}^{3}\sum_{q=1}^{2}a_{pq}n_{il}(\tilde{\theta},\tilde{\omega})_p(-C_l^{-1}\tilde{\omega}_q) \\ &+\sum_{p=1}^{3}a_{p3}n_{il}(\tilde{\theta},\tilde{\omega})_p(-C_l^{-1}\tilde{\kappa}_{kl}) \\ &\qquad\qquad+\sum_{p,q=1}^{3}{}^t a_{pq}n_{il}(\tilde{\theta},\tilde{\omega})_p(-C_i^{-1}\tilde{\theta}_q)\Bigg\} \\ &\qquad\qquad\times|n_{il}(\tilde{\theta},\tilde{\omega})|^{-1}\tilde{P}_{k,0}^l P_l(\tilde{\omega})\Bigg] \\ &{}^tTK(a_t)^{-\frac{1}{2}}\tilde{\beta}_{-2}(\tilde{y}')d\tilde{y}' \\ &+\sum_{j+|\alpha|\ge -1}\int_{\mathbf{R}^2}\rho_j(-s-|n_{il}(\tilde{\theta},\tilde{\omega})||\tilde{y}'|^2/2 \\ &\qquad\qquad - r_{il}(\tilde{\theta},\tilde{\omega}))\tilde{\beta}_j^\alpha(\tilde{y}')\tilde{y}'^\alpha d\tilde{y}', \end{aligned} \tag{3.11}$$

where $\tilde{\beta}_j(\tilde{y}')$ are some C^∞ functions supported near $\tilde{y}'=0$ and $\tilde{\beta}_{-2}(0)=1$. By using Lemma 6.3 and Lemma 6.4 in Soga [11], we show that the leading term of (3.11) is the following form:

$$\begin{aligned} &(2\sqrt{2}\pi)^{-2}C_i^{-\frac{3}{2}}C_l^{-\frac{3}{2}}|n_{il}(\tilde{\theta},\tilde{\omega})|^{-2}\delta^{(1)}(-s-r_{il}(\tilde{\theta},\tilde{\omega})) \\ &\times K(a_t)^{-\frac{1}{2}}|S^1| \\ &\times T\Bigg[P_i(\tilde{\theta})\sum_{k=1}^{2}\Bigg\{\sum_{q=1}^{2}\Big\{a_{3q}(C_l^{-1}\tilde{\omega}_q) + {}^t a_{3q}(C_i^{-1}\tilde{\theta}_q)\Big\} \\ &\qquad + a_{33}(C_l^{-1}\tilde{\kappa}_{kl} + C_i^{-1}\tilde{\theta}_3)\Bigg\}\tilde{P}_{k,0}^l P_l(\tilde{\omega})\Bigg]{}^tT. \end{aligned}$$

Summing over all points a_t, we arrive at the following proposition.

Proposition 3.3. *Let $\omega,\theta\in S^2$. Assume that $|\theta+\omega|$ is sufficiently small, and $n_{il}(\theta,\omega)$ is a regular direction for $\partial\Omega$. Then we have*

$$\begin{aligned} &P_i(\theta)S(s,\theta,\omega)P_l(\omega) \\ &\sim (2\sqrt{2}\pi)^{-2}C_i^{-\frac{3}{2}}C_l^{-\frac{3}{2}}|n_{il}(\tilde{\theta},\tilde{\omega})|^{-2}\delta^{(1)}(-s-r_{il}(\tilde{\theta},\tilde{\omega})) \\ &\times\sum_{t=1}^{M}K(a_t)^{-\frac{1}{2}}|S^1| \\ &\times T\Bigg[P_i(\tilde{\theta})\sum_{k=1}^{2}\Bigg\{\sum_{q=1}^{2}\Big\{a_{3q}(C_l^{-1}\tilde{\omega}_q) + {}^t a_{3q}(C_i^{-1}\tilde{\theta}_q)\Big\} \\ &\qquad + a_{33}(C_l^{-1}\tilde{\kappa}_{kl} + C_i^{-1}\tilde{\theta}_3)\Bigg\}\tilde{P}_{k,0}^l P_l(\tilde{\omega})\Bigg]{}^tT \\ &+\cdots, \end{aligned}$$

where $T=(t_{pq})$ is 3×3 orthogonal matrix and $x = Ty$, $\tilde{\omega} = {}^tT\omega$, $\tilde{\theta} = {}^tT\theta$ and $(\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\cdot\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')})^{-1}\cdot\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\otimes\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{1,0}^l + O(|\tilde{y}'|)$ and $I - (\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\cdot\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')})^{-1}\cdot\nabla\tilde{\varphi}_l^1|_{y_3=\psi(\tilde{y}')}\otimes\nabla\tilde{\varphi}_l^2|_{y_3=\psi(\tilde{y}')} = \tilde{P}_{2,0}^l + O(|\tilde{y}'|)$.

4 Proof of Theorem 2.1

Proof of Theorem 2.1. Note that $P_1(\xi) = \xi\otimes\xi$, $P_2(\xi) = I - \xi\otimes\xi$ and each $\tilde{P}_{k,0}^2 P_2(\tilde{\omega})(k=1,2)$

takes the following form:

$$\bar{P}_1(\tilde{\omega}) := \tilde{P}^2_{1,0}P_2(\tilde{\omega})$$
$$= 2/\bar{A}(\tilde{\omega})\begin{pmatrix} \tilde{\omega}_1^2\tilde{\omega}_3^2 & \tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3^2 & \tilde{\omega}_1|\tilde{\omega}_3|(1-\tilde{\omega}_3^2) \\ \tilde{\omega}_2\tilde{\omega}_1\tilde{\omega}_3^2 & \tilde{\omega}_2^2\tilde{\omega}_3^2 & \tilde{\omega}_2|\tilde{\omega}_3|(1-\tilde{\omega}_3^2) \\ \tilde{\kappa}_{12}\tilde{\omega}_1\tilde{\omega}_3^2 & \tilde{\kappa}_{12}\tilde{\omega}_2\tilde{\omega}_3^2 & \tilde{\kappa}_{12}|\tilde{\omega}_3|(1-\tilde{\omega}_3^2) \end{pmatrix},$$

$$\bar{P}_2(\tilde{\omega}) := \tilde{P}^2_{2,0}P_2(\tilde{\omega}) = 1/\bar{A}(\tilde{\omega})$$
$$\times\begin{pmatrix} \bar{A}(\tilde{\omega})-\tilde{\omega}_1^2 z_+ & -\tilde{\omega}_1\tilde{\omega}_2 z_+ & \tilde{\omega}_1|\tilde{\omega}_3|z_- \\ -\tilde{\omega}_2\tilde{\omega}_1 z_+ & \bar{A}(\tilde{\omega})-\tilde{\omega}_2^2 z_+ & \tilde{\omega}_2|\tilde{\omega}_3|z_- \\ -\tilde{\omega}_3\tilde{\omega}_1 A(\tilde{\omega}) & -\tilde{\omega}_3\tilde{\omega}_2 A(\tilde{\omega}) & (1-\tilde{\omega}_3^2)A(\tilde{\omega}) \end{pmatrix},$$

where $A(\tilde{\omega}) := \tilde{\omega}_1^2+\tilde{\omega}_2^2+\tilde{\omega}_3\tilde{\kappa}_{12}, \bar{A}(\tilde{\omega}) := \tilde{\omega}_1^2+\tilde{\omega}_2^2+|\tilde{\omega}_3|\tilde{\kappa}_{12}$ and $z_\pm := |\tilde{\omega}_3|(\tilde{\kappa}_{12}-\tilde{\omega}_3)\pm 1$. Recall that $n_{i2}(\tilde{\theta},\tilde{\omega})/|n_{i2}(\tilde{\theta},\tilde{\omega})| = (0,0,-1)$, we can rewrite $P_i(\tilde{\theta})\sum_{k=1}^{2}\left[\sum_{q=1}^{2}\{a_{3q}(C_2^{-1}\tilde{\omega}_q)+{}^t a_{3q}(C_i^{-1}\tilde{\theta}_q)\}+a_{33}(C_2^{-1}\tilde{\kappa}_{k1}+C_i^{-1}\tilde{\theta}_3)\right]\tilde{P}^2_{k,0}P_2(\tilde{\omega})$ in the following form:

$$P_i(\tilde{\theta})\sum_{k=1}^{2}\Big\{(a_{31}+{}^t a_{31})\tilde{\omega}_1+(a_{32}+{}^t a_{32})\tilde{\omega}_2 + a_{33}(\tilde{\omega}_3+\tilde{\kappa}_{k2}+C_2|\tilde{n}_{i2}|)\Big\}\bar{P}_k(\tilde{\omega})/C_2. \tag{4.1}$$

Then, calculating each term in (4.1) more carefully, we can obtain

$$P_i(\tilde{\theta})\sum_{k=1}^{2}\begin{pmatrix} 0 & 0 & \lambda+\mu \\ 0 & 0 & 0 \\ \lambda+\mu & 0 & 0 \end{pmatrix}\tilde{\omega}_1\bar{P}_k(\tilde{\omega})$$
$$= (\lambda+\mu)\tilde{\omega}_1\{2(a\otimes q)+\bar{a}\otimes\tilde{p}_{i3}+\bar{c}\otimes\tilde{p}_{i1}\}/\bar{A}(\tilde{\omega}),$$

$$P_i(\tilde{\theta})\sum_{k=1}^{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda+\mu \\ 0 & \lambda+\mu & 0 \end{pmatrix}\tilde{\omega}_2\bar{P}_k(\tilde{\omega})$$
$$= (\lambda+\mu)\tilde{\omega}_2\{2(b\otimes q)+\bar{b}\otimes\tilde{p}_{i3}+\bar{c}\otimes\tilde{p}_{i2}\}/\bar{A}(\tilde{\omega}),$$

$$P_i(\tilde{\theta})\sum_{k=1}^{2}\begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda+2\mu \end{pmatrix}(\tilde{\omega}_3+\tilde{\kappa}_{k2}+C_2|\tilde{n}_{i2}|)\bar{P}_k(\tilde{\omega})$$
$$= \Big\{2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{i2}|)(c\otimes q) + (\tilde{\omega}_3+\tilde{\kappa}_{22}+C_2|\tilde{n}_{i2}|)(\mu(\bar{a}\otimes\tilde{p}_{i1}+\bar{b}\otimes\tilde{p}_{i2}) + (\lambda+2\mu)\bar{c}\otimes\tilde{p}_{i3}\Big\}/\bar{A}(\tilde{\omega}),$$

where

$$a = {}^t(\tilde{\omega}_1\tilde{p}_i^{(13)}+\tilde{\kappa}_{12}\tilde{p}_i^{(11)}, \tilde{\omega}_1\tilde{p}_i^{(23)}+\tilde{\kappa}_{12}\tilde{p}_i^{(21)}, \tilde{\omega}_1\tilde{p}_i^{(33)}+\tilde{\kappa}_{12}\tilde{p}_i^{(31)}),$$

$$\bar{a} = {}^t(\bar{A}(\tilde{\omega})-\tilde{\omega}_1^2 z_+, -\tilde{\omega}_1\tilde{\omega}_2 z_+, \tilde{\omega}_1|\tilde{\omega}_3|z_-)$$

$$b = {}^t(\tilde{\omega}_2\tilde{p}_i^{(13)}+\tilde{\kappa}_{12}\tilde{p}_i^{(12)}, \tilde{\omega}_2\tilde{p}_i^{(23)}+\tilde{\kappa}_{12}\tilde{p}_i^{(22)}, \tilde{\omega}_2\tilde{p}_i^{(33)}+\tilde{\kappa}_{12}\tilde{p}_i^{(32)}),$$

$$\bar{b} = {}^t(-\tilde{\omega}_2\tilde{\omega}_1 z_+, \bar{A}(\tilde{\omega})-\tilde{\omega}_2^2 z_+, \tilde{\omega}_2|\tilde{\omega}_3|z_-)$$

$$c = {}^t(\mu(\tilde{\omega}_1\tilde{p}_i^{(11)}+\tilde{\omega}_2\tilde{p}_i^{(12)})+(\lambda+2\mu)\tilde{\kappa}_{12}\tilde{p}_i^{(13)}, \mu(\tilde{\omega}_1\tilde{p}_i^{(21)}+\tilde{\omega}_2\tilde{p}_i^{(22)})+(\lambda+2\mu)\tilde{\kappa}_{12}\tilde{p}_i^{(23)}, \mu((\tilde{\omega}_1\tilde{p}_i^{(31)}+\tilde{\omega}_2\tilde{p}_i^{(32)})+(\lambda+2\mu)\tilde{\kappa}_{12}\tilde{p}_i^{(33)}),$$

$$\bar{c} = {}^t(-\tilde{\omega}_3\tilde{\omega}_1 A(\tilde{\omega}), -\tilde{\omega}_3\tilde{\omega}_2 A(\tilde{\omega}), (1-\tilde{\omega}_3^2)A(\tilde{\omega})),$$

$$q = {}^t(\tilde{\omega}_1\tilde{\omega}_3^2, \tilde{\omega}_2\tilde{\omega}_3^2, |\tilde{\omega}_3|(1-\tilde{\omega}_3^2)).$$

each $\tilde{p}_i^{(pq)}$ and $\tilde{p}_{il}$ denote (p,q)-entry and lth column of $P_i(\tilde{\theta})$ respectively, and $\tilde{n}_{i2} = n_{i2}(\tilde{\theta},\tilde{\omega})$. Hence, applying the asymptotic expansion derived in the Proposition 3.3, we obtain

$$P_i(\theta)S(s,\theta,\omega)P_2(\omega)$$
$$\sim (2\sqrt{2}\pi)^{-2}C_2^{-\frac{5}{2}}C_i^{-\frac{3}{2}}\delta^{(1)}(-s-r_{i1}(\tilde{\theta},\tilde{\omega}))$$
$$\times\sum_{t=1}^{M}K(a_t)^{-\frac{1}{2}}|S^1|TM(\tilde{\theta},\tilde{\omega})^t T+\cdots,$$

where $M(\tilde{\theta},\tilde{\omega})$ is a 3×3-matrix whose (p,q)-entry is expressed by $m_{pq}(\tilde{\theta},\tilde{\omega})$. As shown above, it is represented in the following form:

$$M(\tilde{\theta},\tilde{\omega})$$
$$= \Big[(\lambda+\mu)\tilde{\omega}_1\{2(a\otimes q)+\bar{a}\otimes\tilde{p}_{i3}+\bar{c}\otimes p_{i1}\} + (\lambda+\mu)\tilde{\omega}_2\{2(b\otimes q)+\bar{b}\otimes\tilde{p}_{i3}+\bar{c}\otimes p_{i2}\}$$
$$+\Big\{2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{i2}|)(c\otimes q) + (\tilde{\omega}_3+\tilde{\kappa}_{22}+C_2|\tilde{n}_{i2}|)\{\mu(\bar{a}\otimes\tilde{p}_{i1}+\bar{b}\otimes\tilde{p}_{i2}) + (\lambda+2\mu)\bar{c}\otimes\tilde{p}_{i3}\}\Big\}\Big]/\bar{A}(\tilde{\omega}).$$

Finally, by considering the case of mode conversion and non-mode conversion case separately, we shall show that the leading term of the right hand side of (3.4) does not vanish. To show it, we prove that $m_{11}(\tilde{\theta},\tilde{\omega}) \neq 0 and m_{33}(\tilde{\theta},\tilde{\omega}) \neq 0$.

Lemma 4.1. *Assume that* $|\tilde{\theta}+\tilde{\omega}|$ *is different from zero and sufficiently small. Then we have* $m_{33}(\theta,\omega) \neq 0$.

Proof. Let $i=1$ (i.e mode-conversion case). According to (3.5), $m_{33}(\tilde{\theta},\tilde{\omega})$ is expressed as follows:

$$\begin{aligned}
&m_{33}(\tilde{\theta},\tilde{\omega})\\
&=\Bigg[(\lambda+\mu)\tilde{\omega}_1\Big\{2(\tilde{\omega}_1\tilde{p}_1^{(33)}+\tilde{\kappa}_{12}\tilde{p}_1^{(31)})|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\\
&+\tilde{\omega}_1|\tilde{\omega}_3|z_-\tilde{p}_1^{(33)}+(1-\tilde{\omega}_3^2)A(\tilde{\omega})\tilde{p}_1^{(31)}\Big\}\\
&+(\lambda+\mu)\tilde{\omega}_2\Big\{2(\tilde{\omega}_2\tilde{p}_1^{(33)}+\tilde{\kappa}_{12}\tilde{p}_1^{(32)})|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\\
&+\tilde{\omega}_2|\tilde{\omega}_3|z_-\tilde{p}_1^{(33)}+(1-\tilde{\omega}_3^2)A(\tilde{\omega})\tilde{p}_1^{(32)}\Big\}\\
&+2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{i2}|)\Big\{\mu(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\
&+(\lambda+2\mu)\tilde{\kappa}_{12}\tilde{p}_1^{(33)}\Big\}|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\\
&+(\tilde{\omega}_3+\tilde{\kappa}_{22}+C_2|\tilde{n}_{i2}|)\{\mu|\tilde{\omega}_3|z_-(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\
&+(\lambda+2\mu)(1-\tilde{\omega}_3^2)A(\tilde{\omega})\tilde{p}_1^{(33)}\}\Bigg]/\bar{A}(\tilde{\omega})\\
&=\Bigg[(\lambda+\mu)|\tilde{\omega}_3|(1-\tilde{\omega}_3^2)\{2(1-\tilde{\omega}_3^2)+z_-\}\tilde{p}_1^{(33)}\\
&\quad+(\lambda+\mu)(1-\tilde{\omega}_3^2)\{2\tilde{\kappa}_{12}|\tilde{\omega}_3|+A(\tilde{\omega})\}\\
&\qquad\times(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\
&+\mu|\tilde{\omega}_3|\Big\{2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{i2}|)(1-\tilde{\omega}_3^2)\\
&\qquad+C_2|\tilde{n}_{i2}|z_-\Big\}(\tilde{\omega}_1\tilde{p}_1^{(31)}+\tilde{\omega}_2\tilde{p}_1^{(32)})\\
&+(\lambda+2\mu)(1-\tilde{\omega}_3^2)\Big\{2(\tilde{\omega}_3+\tilde{\kappa}_{12}+C_2|\tilde{n}_{i2}|)\tilde{\kappa}_{12}|\tilde{\omega}_3|\\
&\qquad+C_2|\tilde{n}_{i2}|A(\tilde{\omega})\Big\}\tilde{p}_1^{(33)}\Bigg]/\bar{A}(\tilde{\omega}).
\end{aligned}$$

By $\tilde{n}_{12}(\tilde{\theta},\tilde{\omega})/|\tilde{n}_{12}(\tilde{\theta},\tilde{\omega})|=(0,0,-1)$, that is,

$$C_2^{-1}\tilde{\omega}_p=C_1^{-1}\tilde{\theta}_p\ (p=1,2),$$
$$C_2^{-1}\tilde{\omega}_3=C_1^{-1}\tilde{\theta}_3-|\tilde{n}_{12}(\tilde{\theta},\tilde{\omega})| \text{ and } |\tilde{\theta}|=1,$$

we can express $m_{33}(\tilde{\theta},\tilde{\omega})$ as a function in $(\tilde{\theta}_1,\tilde{\theta}_2)$:

$$m_{33}(\tilde{\theta}_1,\tilde{\theta}_2)=F(\tilde{\theta}_1,\tilde{\theta}_2)/\bar{A}(\tilde{\omega}(\tilde{\theta}_1,\tilde{\theta}_2)),$$

where

$$\begin{aligned}
&F(\tilde{\theta}_1,\tilde{\theta}_2)\\
&=(\lambda+\mu)C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)|\tilde{\omega}_3(\tilde{\theta})|\\
&\quad\times\{2C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)+z_-\}\{1-(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\}\\
&+(\lambda+\mu)C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)|\tilde{\omega}_3(\tilde{\theta})|\\
&\quad\times\{2\tilde{\kappa}_{12}|\tilde{\omega}_3(\tilde{\theta})|+A(\tilde{\omega}(\tilde{\theta}))\}C_1^{-1}C_2\tilde{\theta}_3(\tilde{\theta})(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\
&+\mu C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)|\tilde{\omega}_3(\tilde{\theta})|\\
&\quad\times\Big\{2(\tilde{\omega}_3(\tilde{\theta})+\tilde{\kappa}_{12}+C_2|\tilde{n}_{12}|)C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\
&\qquad+\{\tilde{\omega}_3(\tilde{\theta})+\tilde{\kappa}_{22}+C_2|\tilde{n}_{12}|\}z_-\Big\}C_1C_2^{-1}\tilde{\theta}_3(\tilde{\theta})\\
&+(\lambda+2\mu)C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\\
&\quad\times\Big\{2(\tilde{\omega}_3(\tilde{\theta})+\tilde{\kappa}_{12}+C_2|\tilde{n}_{12}|)\tilde{\kappa}_{12}\Big\}\tilde{\omega}_3(\tilde{\theta})|\\
&\qquad+C_2|\tilde{n}_{12}|A(\tilde{\omega}(\tilde{\theta}))\Big\}\theta_3^2(\tilde{\theta})\\
&=C_1^{-2}C_2^2(\tilde{\theta}_1^2+\tilde{\theta}_2^2)\tilde{F}(\tilde{\theta}_1,\tilde{\theta}_2).
\end{aligned}$$

Here we note that $|\tilde{\theta}+\tilde{\omega}|\neq 0$ is equivalent to $(\tilde{\theta}_1,\tilde{\theta}_2)\neq(0,0)$. In order to show $m_{33}(\tilde{\theta},\tilde{\omega})\neq 0$, it suffices to show that $\tilde{F}(\tilde{\theta}_1,\tilde{\theta}_2)\neq 0$. Since $\tilde{F}(\tilde{\theta}_1,\tilde{\theta}_2)$ is a C^∞ function near $(\tilde{\theta}_1,\tilde{\theta}_2)=(0,0)$ and $|\tilde{n}_{12}|=C_1^{-1}+C_2^{-1}$, $\tilde{\kappa}_{12}=C_1^{-1}C_2$, $A(\tilde{\omega}(\tilde{\theta}))=-C_1^{-1}C_2$,

$$\begin{aligned}
&\tilde{F}(0,0)\\
&=C_1^{-1}C_2\{(\lambda+\mu)+\mu C_1C_2^{-1}(1+C_1^{-1}C_2)\\
&\qquad+(\lambda+2\mu)(3C_1^{-1}C_2-1)\}\\
&=C_1^{-1}C_2\{\mu C_1C_2^{-1}+3(\lambda+2\mu)C_1^{-1}C_2\}>0
\end{aligned}$$

we can obtain that $\tilde{F}(\tilde{\theta}_1,\tilde{\theta}_2)\neq 0$ provided $|\tilde{\theta}+\tilde{\omega}|$ is different from zero and sufficiently small.

Thus the proof is completed. □

Lemma 4.2. *Assume that* $|\tilde{\theta}+\tilde{\omega}|$ *is different from zero and sufficiently small. Then we have* $m_{11}(\theta,\omega)\neq 0$.

Proof. Let $i = 2$ (i.e non mode conversion case). According to (3.5), $m_{11}(\tilde{\theta}, \tilde{\omega})$ is expressed as follows:

$$
\begin{aligned}
&m_{11}(\tilde{\theta}, \tilde{\omega}) \\
&= \Bigg[(\lambda+\mu)\tilde{\omega}_1 \Big\{ 2(\tilde{\omega}_1 \tilde{p}_2^{(13)} + \tilde{\kappa}_{12}\tilde{p}_2^{(11)})\tilde{\omega}_1\tilde{\omega}_3^2 \\
&\qquad + \{\bar{A}(\tilde{\omega}) - \tilde{\omega}_1^2 z_+\}\tilde{p}_2^{(13)} + (-\tilde{\omega}_3\tilde{\omega}_1)A(\tilde{\omega})\tilde{p}_2^{(11)} \Big\} \\
&+ (\lambda+\mu)\tilde{\omega}_2 \Big\{ 2(\tilde{\omega}_2 \tilde{p}_2^{(13)} + \tilde{\kappa}_{12}\tilde{p}_2^{(12)})\tilde{\omega}_1\tilde{\omega}_3^2 \\
&\qquad + (-\tilde{\omega}_2\tilde{\omega}_1) z_+ \tilde{p}_2^{(13)} + (-\tilde{\omega}_3\tilde{\omega}_1)A(\tilde{\omega})\tilde{p}_2^{(12)} \Big\} \\
&+ 2(\tilde{\omega}_3 + \tilde{\kappa}_{12} + C_2|\tilde{n}_{i2}|)\Big\{ \mu(\tilde{\omega}_1 \tilde{p}_2^{(11)} + \tilde{\omega}_2 \tilde{p}_2^{(12)}) \\
&\qquad + (\lambda+2\mu)\tilde{\kappa}_{12}\tilde{p}_2^{(13)} \Big\}\tilde{\omega}_1\tilde{\omega}_3^2 \\
&+ (\tilde{\omega}_3 + \tilde{\kappa}_{22} + C_2|\tilde{n}_{i2}|)\Big\{ \mu\{(\bar{A}(\tilde{\omega}) - \tilde{\omega}_1^2 z_+)\tilde{p}_2^{(11)} \\
&\qquad + (-\tilde{\omega}_2\tilde{\omega}_1) z_+ \tilde{p}_2^{12}\} \Big\} \\
&\qquad + (\lambda+2\mu)(-\tilde{\omega}_3\tilde{\omega}_1)A(\tilde{\omega})\tilde{p}_2^{(13)} \Bigg] / \bar{A}(\tilde{\omega})
\end{aligned}
$$

Since, in the case of back-scattering, $\tilde{\omega} = (0,0,-1)$ and $\tilde{\theta} = (0,0,1)$, we can derive that $m_{11}(\tilde{\theta}, \tilde{\omega}) = 2\mu + O(|\tilde{\theta} + \tilde{\omega}|)$.

Therefore, by using our assumption that $|\tilde{\theta} + \tilde{\omega}|$ is sufficiently small, we can prove that $m_{33}(\tilde{\theta}, \tilde{\omega}) \neq 0$. □

As shown above, we can prove $m_{11}(\tilde{\theta}, \tilde{\omega}) \neq 0$ and $m_{33}(\tilde{\theta}, \tilde{\omega}) \neq 0$, that is, in each case of mode conversion and non mode cnversion, we show that the leading term of the right-hand side of (3.4) does not vanish. Thus the proof is completed. □

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2